**Finding Antiderivatives** **( Section 7.1 )** **Date:** ______________

Up to this point in the course, one of our major problems has been

**Given a function f, find the derivative f'.**

We needed to be able to <u>find</u> the derivative of a function, before we could <u>use</u> it!

Today, we are going to look at the opposite problem, which also occurs in many applications.

**Given the derivative f', find the function f.**

or we could state this problem as:

**Given a function f, find the function F, such that F' = f.**

**Examples:**

1. If $f'(x) = 20x^4$, then $f(x) =$ ________________

   A question about this answer. Is this a unique answer? ________

   So, we say the antiderivative to $f'(x) = 20x^4$ is $f(x) =$ ____________________.

   This solution represents a "family of functions" for the antiderivative and is called a "general antiderivative". The C is called the "constant of integration" and should be part of every answer to an antiderivative problem. (We will talk about finding "C" later.)

2. If $f(x) = x^4$, then $F(x) =$ ________________________________________

3. If $f(x) = x^{15}$, then $F(x) =$ ________________________________________

4. If $f(x) = \frac{1}{x^2}$ = __________, then $F(x) =$ ________________ = ____________________

5. If $f(x) = \sqrt{x}$ = __________, then $F(x) =$ ________________ = ____________________

Before we write this as a property, let's discuss some new notation.

| <u>Derivative Notation</u> | <u>Antiderivative Notation</u> |
| --- | --- |
| 1. If $f(x) = x^2$, then $f'(x) =$ ________ | 1. If $f(x) = x^2$, then $F(x) =$ ________________ |
| 2. $\frac{d}{dx}\left(x^2\right) =$ ________ | 2. $\int x^2 dx =$ ________________ |
| | This is called an **indefinite integral.** |

So, in general, $\int x^n dx =$ ______________. This is called the **Power Rule for Antiderivatives.**

But be careful! What happens if n = -1? $\int \frac{1}{x} dx = \int x^{-1} dx =$ ______________

______________________________________________

We will handle this later!

Let's find some more antiderivatives, and, from the results, perhaps obtain some properties of indefinite integrals.

**Examples and Properties:** (Assume k is a constant)

1. $\int 7\,dx =$ ______________ **Property**: $\int k\,dx =$ ______________

2. $\int 5x^7 dx =$ ______________ **Property**: $\int (k \cdot f(x))dx =$ ______________

3. $\int \left(2 - 3x^2 + x^5\right)dx =$ ______________ **Property**: $\int \left(f(x) \pm g(x)\right)dx =$ ______________

$=$ ______________

**Note:** Every derivative property has a corresponding antiderivative property!

4. $\int e^x dx =$ ______________

5. $\int e^{4x} dx =$ ______________ **Property**: $\int e^{kx} dx =$ ______________

There is one more antiderivative property we need. What is $\int \frac{1}{x} dx = \int x^{-1} dx$ ? As we have seen, we can't apply t

**Power Rule** because ______________________________

So, we are looking for an antiderivative of $\frac{1}{x}$. Fortunately we know a function whose derivative is $\frac{1}{x}$.

$$\frac{d}{dx}(\_\_\_\_\_\_) = \frac{1}{x}.$$

But, this is only true if $x > 0$, since $\ln x$ is only defined for $x > 0$. What happens if $x < 0$?

If $x < 0$, then $\ln x$ ______________________________, so $\frac{d}{dx}(\ln x)$ ______________________________

But, if $x < 0$, then $\ln(-x)$ is defined, and

$$\frac{d}{dx}(\ln(-x)) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

This means that if $x > 0$, then $\int \frac{1}{x}dx = \ln x + C$, and if $x < 0$, then $\int \frac{1}{x}dx = \ln(-x) + C$.

Can we combine these two so that we are always taking the ln of a positive number? **Yes**, let the answer to the integral be :

$$\ln \_\_\_\_\_\_\_\_\_\_ + C.$$

So, the last antiderivative property says:

**Property:** $\int \frac{1}{x}dx =$ ________________________

One final topic. We know that the **general antiderivative** of

$$f(x) = \frac{3}{x} - 4x \text{ is } F(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

What would we need to find the **specific antiderivative**; i.e. what do we need to find the value of C?

So, if we were given that $f(x) = \frac{3}{x} - 4x$ , and $F(1) = -10$, where F is the antiderivative of f, find the **specific antiderivative** of f(x)?

Therefore, if $f(x) = \frac{3}{x} - 4x$ and $F(1) = -10$, then $F(x) =$ ______________________________

**Example:** If $f(x) = e^{2x} - 9x^2$ and $F(0) = 4$, find F(x), the specific antiderivative of f(x).

**Finding Antiderivatives by the "Substitution Method"** **( Section 7.2 )** **Date:** ____________

In the last class we learned how to find antiderivatives of some functions. We learned that:

1. $\int \frac{1}{\sqrt{x}} dx =$

2. $\int 4e^{5x} dx =$

3. $\int \left(5 + \frac{7}{x}\right) dx =$

Today, we want to increase the types of functions for which we can find antiderivatives.

Before we get into this "method" for finding antiderivatives, we need to understand a few basic concepts.

1. If $y = f(x)$, then $\frac{dy}{dx} =$ ________, or we can write $dy =$ ____________________
   Note: "dy" is called the <u>differential of y.</u>

2. If $w = g(x)$, the $dw =$ ____________________

3. If $\int f(x)dx = h(x) + C$, then $\int f(w)dw =$ ________________________

---

Which antiderivative appears to be easier to find?

1. $\int (x^3 + 1)^6 dx$ 2. $\int 3x^2 (x^3 + 1)^6 dx$

To answer this question, we need to find a function such that:

1. $\frac{d}{dx}($____________$) = (x^3 + 1)^6$ or 2. $\frac{d}{dx}($____________$) = 3x^2 (x^3 + 1)^6$

The result of the second problem is:

$\int 3x^2 (x^3 + 1)^6 dx =$ ______________________________

Note: The only way to find the first antiderivative, $\int (x^3+1)^6\,dx$, is to expand the integrand, and write it as:

$(x^3+1)^6 = x^{18} + 6x^{15} + 15x^{12} + 20x^9 + 15x^6 + 6x^3 + 1$ and then find the antiderivative of each term!

**Another example:** Since $\frac{d}{dx}\left(e^{x^2}\right) =$ ______________, we can write ______________________________

So, our objective is to be able to find an antiderivative of a function such as $\int 2x \cdot e^{x^2}\,dx$.

This results in a technique for finding antiderivatives called the **Substitution Method.** (It is the **Chain Rule** for derivatives in reverse.) For an indefinite integral such as:

$\int 2x \cdot e^{x^2}\,dx$, the idea is to look for an "inside function". For this example the inside function is ______

So, we make a variable substitution and let w = __________. This means dw = ____________________.

We then express the integrand in terms of w.

$\int 2x \cdot e^{x^2}\,dx =$ ____________________

The next step is to find this antiderivative of the function written in terms of w.

And then, we rewrite the answer back in terms of x.

Therefore, $\int 2x \cdot e^{x^2}\,dx =$ __________________________

Finally, you can always check your answer by finding it's derivative!

Let's try another example.

Find $\int 4x^3\sqrt{x^4+5}\,dx$

Let's make a small change to the above problem. Suppose we are asked to find

$$\int x^3\sqrt{x^4+5}\,dx$$

**Some more examples.**

Find the following antiderivatives.

1. $\int \frac{x}{4x^2+1}dx$

2. $\int \frac{1}{\sqrt{3-4x}}dx$

3. $\int e^{-5x} dx$

4. $\int x^2 \sqrt{x^3 - 10}\, dx$

Note: If problem #4 above said $\int x\sqrt{x^3 - 10}\, dx$, we would not be able to find this antiderivative by the **Substitution Method**!

So, to summarize the **Substitution Method** for finding antiderivatives,

1. We make a variable substitution and let w equal an "inside function".
2. We then find dw, and see if the Substitution Method applies.
3. If it does, we then express the integrand in terms of w.
4. We then find this antiderivative of the function expressed in terms of w.
5. Finally, we rewrite the answer back in terms of x.

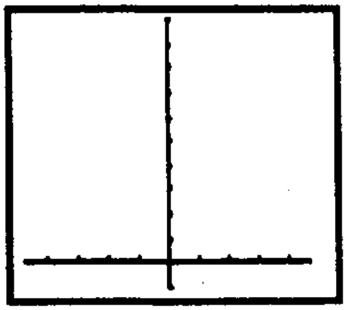
It's now time for you to practice!

**Evaluating Definite Integrals** **( Section 7.3 )** **Date:** ____________

**The Fundamental Theorem of Calculus**

An introductory example:

Find the area of the region bounded by the function $f(x) = 9 - x^2$ and the x-axis. Sketch the region below.



Area of the region = ______________________________

Could we calculate this area with geometry? __________

Actually, we can! Archimedes discovered that the "area under a parabola" can be found by the formula:

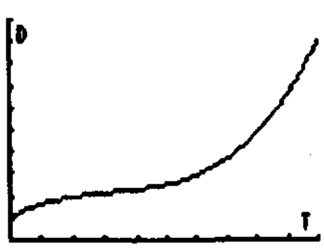
$$\text{Area} = \frac{2}{3} \cdot \text{Base} \cdot \text{Height}.$$

So, the area of the region is ______________________________

But the objective of this lesson, is to discover how to evaluate definite integrals analytically, without our calculator. This result is called the **Fundamental Theorem of Calculus** (FTC).

Let's develop the FTC through an "application".

Honors Physics student Kal Kulis is performing an experiment. He has set a motion detector on a table and it is recording his distance in feet from the motion detector as he is walking away. He collects this data over a 10 second period, enters it into his calculator, and graphs it. The result is shown below.




The graph shows a Distance vs. Time graph. Kal, being also an excellent mathematics student, wants to find a function that models the graph. He decides that the data might be modeled by a cubic function, so he performs a cubic regression on his calculator, and gets a perfect fit!

His function, which represents his distance s(t) from the motion detector at any time t is:

$$s(t) = \frac{1}{10}t^3 - t^2 + 4t + 5$$

Questions:

1. How far did Kal walk over the 10 second time interval? ______________________________

2. Recall that a velocity function is the derivative of a distance function. Sketch a graph of the velocity function below.

3. What is the velocity function for Kal's distance function $s(t) = \frac{1}{10}t^3 - t^2 + 4t + 5$ ?

$v(t) =$ ______________________________ (This is the function we graphed in part 2.)

4. Since the velocity function is a "rate of change" function, how could we use it to determine how far Kal walked over the 10 second time interval?

______________________________________________________________

Let's calculate it (with our calculator) and see what we get! ______________________________

So, here's what we just discovered.

______________________________________________

And, since the velocity v(t) is the derivative of the distance s(t), then the distance s(t) is the ______________ of the velocity v(t).

Let's generalize this result.

If F(x) is the antiderivative of f(x), and we are asked to evaluate $\int_a^b f(x)dx$ , we can do this by writing

$\int_a^b f(x)dx =$ ______________________________________

This is called the **Fundamental Theorem of Calculus**, and we can use it to evaluate definite integrals, if we know how to find the antiderivative of the integrand.

Let's evaluate the definite integral $\int_{-3}^{3}(9 - x^2)dx$ , which we used in the "introductory example" with the FTC and see if we get the same answer (which was 36).

$\int_{-3}^{3}(9 - x^2)dx =$

**Some more examples:**

Evaluate the following definite integrals with the FTC.

1. $\int_4^9 \sqrt{x}\,dx$

2. $\int_1^2 \frac{3}{x^4}dx$

3. $\int_0^1 \left(5e^x + 6x\right)dx$

4. $\int_1^e \frac{1}{t}dt$

In conclusion, The **Fundamental Theorem of Calculus** states:

If $f(x)$ is a continuous function on the interval $[a, b]$, and $F(x)$ is the antiderivative of $f(x)$,

then $\int_a^b f(x)dx =$ ______________________________

We have actually used this earlier, when we did **Section 5.4: Interpreting the Definite Integral.**

**Example #1:** Assume $f(t) = 60\sqrt{t}$ gives the rate of change of the population of a city, in people per year, at time . years since 1990. If the population of the city is 3000 in 1990, what is the population in 2000?

**The FTC with the Substitution Method** **( Section 7.3 continued )** **Date: __________**

The last two sections presented in the course were:

1. Finding antiderivatives with the Substitution Method.
2. Evaluating definite integrals using the Fundamental Theorem of Calculus

Today, the last day(!), we are going to combine these two topics. In other words, how do we use the FTC to evaluate a definite integral if it is necessary to use the Substitution Method to find the antiderivative.

There are two methods that we can use to do this. Let's look at an example.

**Example:**

Evaluate: $\int_0^1 x\left(x^2+1\right)^3 dx$

**Method #1:** To evaluate this definite integral using the FTC, we first need to find an antiderivative of the function $f(x)=x\left(x^2+1\right)^3$, which requires the Substitution Method. So, let's first treat the problem as an <u>indefinite integral</u> (antiderivative), and find:

$$\int x\left(x^2+1\right)^3 dx =$$

Now, that we have the antiderivative, we can find the definite integral.

$$\int_0^1 x\left(x^2+1\right)^3 dx =$$

Note: Check the answer with your calculator.

**Method #2**: Once we have established the "change of variable" in the definite integral, and we have written the antiderivative in terms of the new variable, we can also rewrite the limits of integration in terms of the new variable.

$$\int_0^1 x\left(x^2+1\right)^3 dx$$

With this method, we do not convert the antiderivative back to the original variable.

Let's try some more examples to get some practice.

**Examples and Practice Problems:**

Analytically, evaluate the following definite integrals. You can then check answers with your calculator.

1. $\int_0^2 x^2\sqrt{1+x^3}\,dx$

2. $\int_0^4 \frac{dx}{\sqrt{2x+1}}$

3. $\int_1^2 \frac{dx}{(3-5x)^2}$

4. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

5. $\int_0^3 \frac{6x}{x^2+1}dx$

6. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = 460e^{1.2t}$ bacteria per hour. How many bacteria will there be after 3 hours?